

MAS 117

Introduction to Algebra

Solutions 10

January 2007

- 1** Follow the algorithm given in lectures: Open a bracket and put the first number 1. Then

$$1 \mapsto 4 \mapsto 7 \mapsto 1,$$

so complete the cycle $(1, 4, 7)$. We are not finished: open another bracket with 2 and complete the cycle $(2, 10)$; then the cycle $(3, 9, 6, 5)$; and finally the cycle (8) (which we can omit). So

$$g = (1, 4, 7)(2, 10)(3, 9, 6, 5).$$

(You will not be wrong if you include the (8) .)

The order of a permutation is the least common multiple of its cycle lengths. In this case, the lcm of 3, 2, 4 and 1 is 12.

- 2** For h^{-1} , just read the cycles backwards:

$$h^{-1} = (4, 7, 9, 5, 1)(6, 3, 2)(10, 8).$$

For gh , you do it like this. Open a bracket and put in the first number 1. Then

$$1 \mapsto 4 \text{ (by } g) \mapsto 1 \text{ (by } h),$$

so immediately close the bracket. Open another bracket and put 2. Then

$$2 \mapsto 10 \text{ (by } g) \mapsto 8 \text{ (by } h),$$

so write 8 in the bracket. Then

$$8 \mapsto 8 \text{ (by } g) \mapsto 10 \text{ (by } h),$$

so write 10. And so on. We find that

$$gh = (2, 8, 10, 3, 7, 5, 6, 9)$$

(with optionally the two cycles (1) and (4)).

For $h^{-1}gh$, multiply the two permutations just found by the same procedure:

$$h^{-1}gh = (1, 4, 5)(2, 9, 6, 7)(3, 8)$$

(with optional (10)).

3 (a) The order of S_3 is $3! = 6$ (the number of permutations of $\{1, 2, 3\}$).

(b) The identity has order 1. The three transpositions $(1, 2)$, $(1, 3)$ and $(2, 3)$ each have order 2. The other two elements $(1, 2, 3)$ and $(1, 3, 2)$ each have order 3.

(c) By Lagrange's Theorem, elements of a subgroup of order 2 must have order 1 or 2, and the identity must obviously be included. So there are three possibilities: $\{e, (1, 2)\}$, $\{e, (1, 3)\}$, and $\{e, (2, 3)\}$. It is easy to see that each is a subgroup. Most easily: for any element g of a group G , we know that $\langle g \rangle$, the subgroup generated by g (which consists of all the powers g^n for $n \in \mathbb{Z}$) is a subgroup, and its order is equal to the order of g . So, for example, $\langle (1, 2) \rangle = \{e, (1, 2)\}$ is a subgroup of order 2.

Similarly a subgroup of order 3 must have elements of orders 1 or 3, and there is only one possibility: $\{e, (1, 2, 3), (1, 3, 2)\}$ (which again is a subgroup, as it is $\langle (1, 2, 3) \rangle$).

4 We saw in the lectures that a cycle of length m is the product of $m - 1$ transpositions. If g is a permutation of odd order, then each of its cycles must have odd length, since the order is the least common multiple of the cycle lengths. So each cycle is a product of an even number of transpositions; and so g is a product of an even number of transpositions, since a sum of even numbers is even.

The converse is false. The converse would state "any even permutation has odd order". But the permutation $(1, 2)(3, 4)$ is the product of two transpositions, so is an even permutation, but its order is 2, which is even.